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PLANE STRUCTURES IN THERMAL RUNAWAY

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ABSTRACT

We consider the problem

(1) $u_t = u_{xx} + e^u$ when $x \in \mathbb{R}$, $t > 0$,

(2) $u(x,0) = u_0(x)$ when $x \in \mathbb{R}$,

where $u_0(x)$ is continuous, nonnegative and bounded. Equation (1) appears as a limit case in the analysis of combustion of a one-dimensional solid fuel. It is known that solutions of (1), (2) blow-up in a finite time T, a phenomenon often referred to as thermal runaway. In this paper we prove the existence of blow-up profiles which are flatter than those previously observed. We also derive the asymptotic profile of $u(x,T)$ near its blow-up points, which are shown to be isolated.

1. Introduction

Consider the semilinear parabolic equation

$$u_t - u_{xx} = e^u; \qquad x_0 \in (a,b), \quad t > 0, \tag{1.1}$$

where $-\infty \le a < b \le +\infty$. Equation (1.1) is one of the simplest models arising from combustion theory. Indeed, it is well known (cf. for instance [BE], Chapter I) that thermal reaction of a one-dimensional solid fuel can be described by the system

$$T_t = T_{xx} + \delta\varepsilon c \exp\left[\frac{T-1}{\varepsilon T}\right], \tag{1.2a}$$

$$c_t = -\varepsilon\delta\Gamma c \exp\left[\frac{T-1}{\varepsilon T}\right] \tag{1.2b}$$

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where T and c are respectively the fuel temperature and concentration, and δ, ε, Γ are positive constants. If we assume ε to be small, $0<\varepsilon\ll 1$, and look for solutions in the form

$$T = 1+\varepsilon u+\cdots$$
$$c = 1-\varepsilon C_1+\cdots$$

we obtain, to the first order, that u satisfies (1.1). Of course, (1.1) and (1.2) are to be complemented with suitable initial and boundary conditions to yield well-posed mathematical problems. In any case, a major feature of (1.1) is that, under suitable assumptions on the size of the set (a,b), positive solutions may become unbounded in a finite time, i.e, there exists $T>0$ such that

$$\lim_{t\uparrow T}\sup(\sup_{(a,b)} u(x,t)) = +\infty.$$

We then say that blow-up or thermal runaway occurs at $t=T$. If this is the case, we say that $x=x_0$ is a blow-up point of $u(x,t)$ if there exist sequences $\{x_n\}$, $\{t_n\}$, such that $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} t_n = T$ and $\lim_{n\to\infty} u(x_n,t_n)=+\infty$.

In recent years, much attention has been devoted to the question of determinig when do solutions of (1.1) exhibit thermal runaway and, in such a case, where are the blow-up points located. For instance, conditions under which blow-up occurs at a single point have been obtained in [W],[FM]. A further natural question is: what is the asymptotic behaviour of solutions near blow-up points as thermal runaway is approached. An important step towards ascertaining this point was given by J.W. Dold in [D]. Using matched asymptotic expansions techniques, he formally derived an expansion for a solution of (1.1) which was assumed to blow-up at $x=0, t=T$. To describe his result, let us introduce auxiliary variables as follows:

$$\begin{aligned} u(x,t) &= -\ln(T-t)+\psi(z,\tau),\\ z &= \frac{x}{((T-t)|\ln(T-t)|)^{1/2}},\\ \tau &= -\ln(T-t). \end{aligned} \tag{1.3}$$

Then Dold claimed that

$$\begin{aligned} \psi(z,\tau) = &-\ln\left(1+\frac{z^2}{4}\right)-\frac{5}{8}\cdot\frac{\ln\tau}{\tau}\cdot\frac{z^2+4}{1+z^2/4}\\ &+\frac{1}{\tau}\left(\frac{1}{2}+\frac{z^2/4}{1+z^2/4}\left(C-\ln\left(1+\frac{z^2}{4}\right)\right)\right)+o\left(\frac{1}{\tau}\right) \end{aligned} \tag{1.4}$$

for some real constant C, uniformly for z in bounded sets.

A first rigorous asymptotic result is

$$\lim_{t\uparrow T}\left(u(x(T-t)^{1/2},t)+\ln(T-t)\right)=0, \tag{1.5}$$

uniformly on sets $|x|<R$ with $R>0$,

which holds under suitable assumptions on the data for the Cauchy–Dirichlet problem corresponding to (1.1) (cf [BBE]), and is also verified by solutions of the Cauchy problem, if the initial value is continuous, nonnegative and bounded (cf. for instance [HV1]). Recently, A. Bressan proved in [B1] that, if one considers the boundary value problem

$$u_t = u_{xx} + e^u \qquad \text{when } x_0 \in [-1,1], \quad t>0, \tag{1.6a}$$

$$u(\pm 1, t) = 0 \qquad \text{when } t>0 \tag{1.6b}$$

$$u(x,0) = u_0(x) \qquad \text{when } -1 \le x_0 \le 1, \tag{1.6c}$$

then there exist initial values $u_0(x)$ such that the corresponding solution $u(x,t;u_0(x))$ of (1.6) blows up at $x=0$, $t=T$, and (1.4) holds. Furthermore, the asymptotics near $x=0$, $t=T$ was shown to be stable under small perturbations of such data $u_0(x)$.

These results, however, do not preclude the existence of solutions exhibiting different blow-up behaviour, although they strongly suggest that, if such solutions exist, they should be of an unstable character. To our knowledge, the possibility of asymptotics other than (1.4) was first suggested in [GHV], by means of (formal) perturbation techniques. We then were able to make such results rigorous in [HV1] for the case of the Cauchy problem

$$u_t = u_{xx} + e^u; \qquad -\infty < x < +\infty, \quad t>0, \tag{1.7a}$$

$$u(x,0) = u_0(x); \qquad -\infty < x < +\infty. \tag{1.7b}$$

More precisely, let $\bar{x}$ be a real number, and let $\psi(y,\tau), y, \tau$ be given by

$$u(x,t) = -\ln(T-t) + \psi(y,\tau); \quad y = \frac{x-\bar{x}}{\sqrt{T-t}}, \quad \tau = -\ln(T-t). \tag{1.8}$$

We proved in [HV1] the following result:

THEOREM A: *Assume that $u_0(x)$ is continuous, nonnegative and bounded. Suppose also that $u(x,t)$ solves (1.7), blows up at $x = \bar{x}$, $t = T$, and is such that $u(x,t) \neq -\ln(T-t)$. Then, either*

$$\psi(y,\tau) = -\frac{(4\pi)^{1/4}}{\sqrt{2}} \cdot \frac{H_2(y)}{\tau} + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to \infty, \tag{1.9a}$$

or

There exist $C \neq 0$ and $m > 3$ such that

$$\psi(y,\tau) = Ce^{(1-\frac{m}{2})\tau} H_m(y) + o\left(e^{(1-\frac{m}{2})\tau}\right) \quad \text{as } \tau \to \infty, \tag{1.9b}$$

where convergence takes place in $C^{k,\alpha}_{\text{loc}}$ for any $k \geq 1$ and $\alpha \in (0,1)$, and

$$H_m(y) = c_m \tilde{H}_m\left(\frac{y}{2}\right), \quad c_m = \left(2^{m/2}(4\pi)^{1/4}(m!)^{1/2}\right)^{-1}, \tag{1.9c}$$

$\tilde{H}_m(s)$ being the standard m^{th} Hermite polynomial.

We have obtained in [HV1] a result similar to Theorem A for the problem

$$u_t = u_{xx} + u^p; \qquad -\infty < x < +\infty, \quad t > 0, \quad p > 1, \tag{1.10a}$$

$$u(x,0) = u_0(x); \qquad -\infty < x < +\infty, \tag{1.10b}$$

where $u_0(x)$ is as before. See also [FK] for related results. On the other hand, in the recent work [BB], J. Bebernes and S. Bricher have used the techniques in [FK], [HV1], [HV3] to show that (1.9a) holds for radial solutions of the higher-dimensional version of (1.7), provided that $u_0(x) \in C^2$, radially decreasing, and

$$\Delta u_0(x) + e^{u_0(x)} \geq 0, \tag{1.11}$$

so that $u_t \geq 0$ at any time $t > 0$. Here we shall prove

THEOREM 1: (a) *Assume that $u_0(x)$ is continuous, nonnegative and bounded, and has a single maximum at some point $x = x_0$. Then the solution $u(x,t)$ of (1.7) blows up at a single point $x = \bar{x}$ at a time $t = T < +\infty$, and* (1.9a) *holds.*

(b) *There exist a continuous, nonnegative and bounded function $\bar{u}_0(x)$ and a constant $C > 0$, such that the corresponding solution of* (1.7), *$\bar{u}(x,t)$, blows up at $x = 0$ at some time $T > 0$, and (1.9b) holds with $m = 4$, i.e,*

$$\psi(y,\tau) = Ce^{-\tau} H_4(y) + o(e^{-\tau}) \qquad \text{as } \tau \to \infty \quad \text{in } C^{k,\alpha}_{\text{loc}}, \tag{1.12}$$

for any $k \geq 1$ *and* $\alpha \in (0,1)$.

Concerning Theorem 1, some remarks are in order. Firstly, in part (a) $u_0(x)$ is not supposed to be symmetric, and assumption (1.11) is not required here. On the other hand, our approach relies heavily on the classification of singularities given in Theorem A, and makes use of the scaling properties of (1.7a) and various *a priori* estimates. Of those, some are of independent interest, as, for instance, the bounds given in Lemmas 2.3 and 2.5 below; in particular, this last enables us to dispense with condition (1.11) here. We shall also employ the following result, which will be derived in the course of the proof.

THEOREM 2: *Let* $u_0(x)$, $u(x,t)$ *and* T *be as in Theorem 1, and assume that* $u(x,t) \not\equiv -\ln(T-t)$. *Then any blow-up point of* $u(x,t)$ *is isolated. Moreover, if* x *is any such point, we have:*

If (1.9a) *is satisfied, then*

$$\lim_{x \to \bar{x}} \left(u(x,T) + \ln\left(\frac{|x-\bar{x}|^2}{8|\ln(x-\bar{x})|} \right) \right) = 0. \tag{1.13a}$$

If (1.9b) *is satisfied, then*

$$\lim_{x \to \bar{x}} \left(u(x,T) + \ln\left(Cc_m |x-\bar{x}|^m \right) \right) = 0. \tag{1.13b.}$$

where C, m, *are as in* (1.9b), *and* c_m *is given in* (1.9c).

We should point out that a final-time analysis analogous to (1.13) also holds for (1.10); cf [HV3]. The fact that the set of initial values for which (1.13a) holds is nonempty has been recently proved in [B2]. Also, (1.13a) has been recently extended in [BB] for radial solutions of the higher-dimensional version of (1.7) under assumption (1.11), whereas the fact that blow-up points are isolated was proved in [CM] for some boundary-value problems associated to (1.7a). Notice that (1.12) and (1.13b) describe a behaviour near the singularity which is flatter than that corresponding to (1.9a), (1.13a). This motivates the term *plane structures* in the title of this paper. On the other hand, the existence of flat behaviours alike to (1.12) has been previously obtained by us for problem (1.10). (cf. Theorem 3.2 in [HV2]). However, the proof of (1.12) here will exhibit some differences with respect of that of the corresponding result for (1.10). In some cases, these arise from the different scaling properties of (1.7a) and (1.10a), but we shall also use here Theorem 2 to show that, if $u_0(x)$ has exactly two peaks, the corresponding solution of (1.7) has one or two blow-up points. This was shown in [CF] for the case of homogeneous Dirichlet problems in bounded domains, by means of a comparison device introduced in [FM] whose use is avoided here.

2. The proofs

2.1 PRELIMINARIES. Before going into the details, let us describe heuristically the way in which Theorem 1 will be obtained. Let ε, h, R be positive numbers, and consider the functions

(2.1a) $\varphi(x;h,\varepsilon) = \frac{h}{\varepsilon}(\varepsilon - |x|_0)_+$, where $s_+ = \max\{s, 0\}$,

(2.1b) $\bar{u}(x,t) \equiv \bar{u}(x,t;h,\varepsilon)$, defined as the solution of (1.7a) satisfying $\bar{u}(x,0) = \varphi(x;h,\varepsilon)$ in $\mathbb{R}$,

(2.1c) $u_R(x,t) \equiv u_R(x,t;h,\varepsilon)$, defined as the solution of (1.7a) with initial value $u_R(x,0) = \varphi(x-R;h,\varepsilon)$,

(2.1d) $u_R^{\pm}(x,t;h,\varepsilon)$, defined as the solution of (1.7a) with initial value $u_R^{\pm}(x,0) = \varphi(x-R;h,\varepsilon) + \varphi(x+R;h,\varepsilon)$.

We then proceed as follows. By the results of Theorem A, in the neighborhood of any blow-up point x, the space structure of $\psi(y,\tau)$ (cf (1.8)) is described by a Hermite polynomial $H_m(y)$ with $m \geq 2$. When $R = 0$, $u_R^{\pm}(x,t)$ is symmetric with respect to the origin and has a single maximum at $x = 0$. Therefore, as the number of maxima of solutions of parabolic equations cannot increase in time (cf. for instance [A], [AF]), (1.9a) should occur in this case, since $H_2(y)$ has a single extremum at $y = 0$. By continuity, we expect that the same will happen to $u_R^{\pm}(x,t)$ when $R > 0$ is sufficiently small. On the other hand, if $R > 0$ is large enough, we expect thermal runaway to take place at two distinct points $x = \pm x_0$. Comparing the cases $R \sim 0$ and $R \gg 0$, we guess that in both situations $u_R^{\pm}(x,t)$ would have two maxima for small times. However, while they should coalesce into a single one before the blow-up time when R is small, they will remain separated until thermal runaway occurs for R large. This last case gives raise to two simultaneous explosions of type (1.9a) at two different points. We then expect that an intermediate value R^* will exist, such that $u_{R^*}(x,t)$ will have two maxima for any time t prior to blow-up, when they will collapse into a single one. This would correspond to $\psi(y,\tau)$ having a space structure given by $H_4(y)$ (which has exactly two extrema), thus showing the result. These are the ideas behind the proof of the corresponding result for the power-like case (1.10) in [HV2].

To carry out such a program, we have introduced some changes here with respect to [HV2]. Specifically, the main steps in the proof are the following:

(i) For any $R \geq 0$, the blow-up points of $u_R^{\pm}(x,t)$ remain in a bounded set (which depends on R). This may be done by adapting the energy method in [GK2], Thm 3.5, but we have selected a different method here, which makes use of Lemma 2.5 below.

(ii) Blow-up occurs at isolated points.This is proved as a consequence of Theorem 2.

(iii) For any $R \geq 0$, blow-up occurs at one or two points. We obtain such result as a consequence of (i), Theorem A and the fact that, in (1.9b), m is even and $m \geq 4$ (cf. Proposition 2.4 below), since the number of maxima of solutions of (1.1) is nonincreasing (cf [A], [AF]).

(iv) If $R \gg 0$, there is no blow-up at $x = 0$, a fact which is obtained again by means of Lemma 2.5 .

(v) The blow-up time of $u_R^{\pm}(x,t)$ as well as the location of its blow-up points depend continuously on R; cf. Lemmas 2.2 and 2.6 below.

2.2 THE BASIC TOOLS. We begin by recalling a particular subsolution of (1.7a) which will be repeatedly used in what follows. Let $f(x)$ be continuous, nonnegative and bounded, and denote by $S(t)f$ the solution of the heat equation in the whole line with initial value $f(x)$. Then the function,

$$u(x,t;f) = -\ln(\exp(-S(t)f(x)) - t), \tag{2.1a}$$

satisfies, for some $T = T(f) > 0$,

$$u_t \leq u_{xx} + e_0^u \qquad \text{when } x \in \mathbb{R}, \quad t \in (0,T). \tag{2.1b}$$

$$u(x,0) = f(x) \qquad \text{when } x \in \mathbb{R}. \tag{2.1c}$$

We shall also need the following lower bounds:

LEMMA 2.1: *Let u_0, u, T and ψ be as in the statement of Theorem 1. We then have,*

If (1.9a) *holds,*

$$\liminf_{t \uparrow T} \left(u\left(\xi \Big((T-t)|\ln(T-t)| \Big)^{1/2}, t \right) + \ln(T-t) \right) \geq -\ln\left(1 + \frac{\xi^2}{4}\right), \tag{2.2a}$$

uniformly on sets $|\xi| \leq R$ with $R > 0$.

If (1.9b) *holds,*

$$\text{(2.2b)} \qquad \liminf_{t \uparrow T} \left(u\left(\xi \left((T-t) \right)^{1/m}, t \right) + \ln(T-t) \right) \geq -\ln\left(1 + Cc_m \xi^m \right),$$

uniformly on sets $|\xi| \leq R$ with $R > 0$, where C, m, are as in (1.9b), *and c_m is given in* (1.9c).

Proof: It is similar to those of Lemmas 6.1 in [HV1] and 2.1 in [HV2], the only differences arising from the use of (2.1a) to replace the subsolutions employed in those papers. ∎

We shall denote henceforth the blow-up time of $u_R^{\pm}(x,t)$ by $T_R(\varepsilon,h)$ or T_R, according to the context. We next prove

LEMMA 2.2: *$T_R(\varepsilon,h)$ is a continuous function of R, h and ε.*

Proof: For simplicity, we shall only prove continuity with respect to h, the other cases being similar. The fact that

$$\lim_{h_n \to h} \inf T_R(\varepsilon, h_n) \geq T_R(\varepsilon, h)$$

is rather classical; see for instance the argument in [H], Th. 3.4.1. However, since the proof is short, we shall sketch it here for completeness. Clearly, it suffices to show that, for any $\sigma > 0$ small enough, there exists n_0 such that $u(x,t;h_n)$ is bounded for $t \leq T(h) - \sigma$, provided that $n \geq n_0$. By assumption, we have that $u(x,t;h) \leq M$ for some $M = M(\sigma)$, whenever $t \leq T(h) - \sigma$. On the other hand, there exists $L = L(\sigma)$ such that

$$|\exp(u(x,t;h_n)) - \exp(u(x,t;h))| \leq L|u(x,t,;h_n) - u(x,t;h)|,$$

as far as $u(x,t;h_n) \leq 2M$ and $t \leq T(h) - \sigma$.

Using Kato's inequality ($\Delta|f| \geq \Delta f \cdot \text{sgn} f$ in D', for $f \in L^1_{\text{loc}}$ with $\Delta f \in L^1_{\text{loc}}$), we then obtain for (1.7a) that

$$|u(x,t;h_n) - u(x,t;h)| \leq e^{Lt}(\sup_{x \in \mathbb{R}}|u(x,0;h_n) - u(x,0;h)|$$

therefore $u(x,t;h_n) \leq 2M$ for $t \leq T(h) - \sigma$ if n is large enough, and the result follows.

Suppose now that $u(x,t;h)$ blows up at $x=0$ as indicated in (2.2a). We then have that, for any $\delta > 0$ and $\theta > 0$, there exists $K = K(\theta,\delta)$ such that

$$\begin{aligned} &u(x,t;h) \geq -\ln(T(h)-t) - K, \quad \text{uniformly on sets} \\ &|x| \leq \theta((T(h)-t)|\ln(T(h)-t|)^{1/2} \quad \text{with } T(h)-t \leq \delta. \end{aligned} \tag{2.4}$$

Take now a sequence $\{h_n\}$ with $\lim_{n\to\infty} h_n = h$. From (2.4) with $t = T(h)-\delta$ and standard continuous dependence results, we deduce that, if n is large enough,

$$u(x, T(h)-\delta; h_n) \geq -\ln\delta - 2K \quad \text{if } |x| \leq \theta\delta^{1/2}|\ln\delta|^{1/2}. \tag{2.5}$$

Consider now the scaled functions

$$u_{\delta,n}(x,t) = \ln\delta + u(\delta^{1/2}x, T(h)-\delta+\delta t; h_n). \tag{2.6}$$

It is immediate to see that, for any $\delta > 0$ and $n = 1, 2, \dots,$ $u_{\delta,n}$ solves (1.7a), whereas for large n we have

$$\begin{aligned} &u_{\delta,n}(x,0) \geq -2K \quad \text{if} \quad |x| \leq \theta|\ln\delta|^{1/2}; \\ &u_{\delta,n}(x,0) \geq \ln\delta \quad \text{if} \quad |x| > \theta|\ln\delta|^{1/2}. \end{aligned}$$

Therefore

$$u_{\delta,n}(x,t) \geq w_\delta(x,t) \tag{2.7}$$

where $w_\delta(x,t)$ solves (1.7a) with initial value $w_\delta(x,0) = -2K$ if $|x| \leq \theta|\ln\delta|^{1/2}$, $w_\delta(x,0) = \ln\delta$ if $|x| > \theta|\ln\delta|^{1/2}$. Furthermore, by (2.1),

$$u_\delta(x,t) \geq -\ln(\exp(-S(t)w_\delta(x,0)) - t).$$

We next notice that

$$\lim_{\delta\downarrow 0} S(t)w_\delta(0,t) = -2K, \qquad \text{uniformly for } t \text{ bounded.} \tag{2.8}$$

To see this, we write

$$\begin{aligned} S(t)w_\delta(0,t) &= \frac{-2K}{\sqrt{4\pi t}} \int_{|\xi|\leq|\ln\delta|^{1/2}} e^{-\frac{\xi^2}{4t}}\,d\xi + \frac{\ln\delta}{\sqrt{4\pi t}} \int_{|\xi|\geq|\ln\delta|^{1/2}} e^{-\frac{\xi^2}{4t}}\,d\xi \\ &\equiv I_1(\delta) + I_2(\delta). \end{aligned}$$

It then follows that $\lim_{\delta\downarrow 0} I_1(\delta) = -2K$, whereas for $t \le T^* < +\infty$,

$$|I_2(\delta)| \le C|\ln\delta|^{1/2}\delta^{\theta^2/C} \qquad \text{for some } C = C(T^*),$$

whence (2.8). We now select $\delta > 0$ small enough and $\theta > 0$ fixed. Taking $T^* = 3e^{2K}$, (2.8) yields then that $S(t)w_\delta(0,t) \le 2e^{2K}$ for $t \le T^*$, and $w_\delta(x,t)$ blows up in a time $T(w)$ such that $T(w) = O(1)$ as $\delta \downarrow 0$. Recalling (2.6), (2.7), we obtain

$$\lim_{n\to\infty} \sup T(h_n) \le T(h). \tag{2.9}$$

Putting together (2.3) and (2.9), the Lemma follows under our current assumptions. The proof of the remaining cases is similar and will be omitted. ∎

We next obtain an a priori bound for blowing up solutions.

LEMMA 2.3: *Let $u(x,t)$ be a solution of (1.7) with $u_0(x)$ continuous, nonnegative and bounded, which blows up at $t = T$. Then there exists a constant C depending on $|u_0|_\infty$ and T, such that*

$$u(x,t) \le -\ln(T-t) + \frac{C}{|\ln(1-\frac{t}{T})|}. \tag{2.10}$$

Proof: Let $u_1(x,t)$, $u_2(x,t)$ be two real functions. Following [GP], we say that r is an intersection point of u_1, u_2 at time $t = t_0$ if $u_1(r,t_0) = u_2(r,t_0)$, and $(u_1(x,t_0) - u_2(x,t_0))$ changes sign when x passes through the value $x = r$. Assume now that $T = 1$, and $u_0(x) \in C^1(\mathbb{R})$ with $\max_{x\in\mathbb{R}} u_0(x) + \max_{x\in\mathbb{R}} |u_0'(x)| \le K < +\infty$. Let $\varphi(x;h,\varepsilon)$ be the function defined in (2.1a). Clearly, by selecting ε and h in a suitable way, we may impose that $u_0(x)$ and $\varphi(x;h,\varepsilon)$ have exactly two intersections. Furthermore, by varying ε or h, we may also obtain that the blow up time of $\bar{u}(x,t)$ given in (2.1b) be equal to one. Since the number of intersections between u and $\bar{u}$ cannot increase in time (cf. for instance [GP], [HV2]) and both solutions blow-up at $t = 1$, we deduce that

$$u(0,t) \le \bar{u}(0,t) \quad \text{for any } t \in (0,1)$$

and, by (1.9),

$$u(0,t) \le \bar{u}(0,t) \le -\ln(1-t) + \frac{B}{|\ln(1-t)|},$$

for some $B > 0$. We now repeat the previous argument with $\varphi(x; h, \varepsilon)$ replaced by $\varphi(x - x_0; h, \varepsilon)$, to deduce that (2.10) holds when $T = 1$. The case where $T \neq 1$ is obtained by scaling. Indeed, if $u(x,t)$ solves (1.7a), then for any $\lambda > 0$,

$$u_\lambda(x,t) = \ln \lambda + u(\sqrt{\lambda}x, \lambda t) \tag{2.11}$$

is also a solution of (1.7a). Setting $\lambda = T$, the general form of (2.10) follows in our case. Finally, the regularity assumptions on $u_0(x)$ can be dispensed with, since they hold for any $t > 0$ by standard parabolic theory. ∎

As a first application of Lemma 2.3, we now derive the following result which completes Lemma 2.1 by extending the convergence stated in Theorem A to larger regions.

PROPOSITION 2.4: *Let u_0, u, T and ψ be as in the statement of Theorem 1. We then have:*

If (1.9a) *holds,*

$$\lim_{t \uparrow T} \left\{ u\left(\xi((T-t)|\ln(T-t)|)^{1/2}, t \right) + \ln(T-t) \right\} = -\ln\left(1 + \frac{\xi^2}{4}\right), \tag{2.12a}$$

uniformly on sets $|\xi| \leq R$ with $R > 0$.

If (1.9b) *holds,*

$$\lim_{t \uparrow T} \left\{ u\left(\xi(T-t)^{\frac{1}{m}}, t \right) + \ln(T-t) \right\} = -\ln\left(1 + C c_m \xi^m\right) \tag{2.12b}$$

uniformly on sets $|\xi| \leq R$ with $R > 0$ where $C > 0$, m is an even number, $m \geq 4$, and c_m is given in (1.9c).

Proof: Once (2.2) and (2.10) have been obtained, the proof is entirely similar to that in [HV1], Section 6 and [HV2], Section 2 for the power-like case (1.10). The fact that now $C > 0$ and m is even, $m \geq 4$ (instead of the weaker conditions in (1.9b)) follows at once, since otherwise (2.10) would not hold. ∎

Remark: Estimate (2.12a) has been obtained in [HV1] under the assumptions that $u_0(x)$ has a single maximum at some point $x = x_0$ and is symmetric with respect to x_0. ∎

Let T, a, b, ε and μ be positive real numbers such that $a < b$, and consider the boundary value problem

$$u_t = u_{xx} + e^u \qquad \text{when } a < x < b, 0 < t < T, \tag{2.13a}$$

$$u(a,t) = u(b,t) = f(t) \quad \text{for } 0 < t < T, \tag{2.13b}$$

$$u(x,0) = g(x) \qquad \text{for } a \leq x \leq b, \tag{2.13c}$$

where

$$f(t) = -\ln(T-t) + \varepsilon, \qquad 0 < t < T, \tag{2.14a}$$

$$g(x) = \begin{cases} -\ln T + \varepsilon, & \text{if } a \le x \le \frac{b-a}{4} \text{ or } \frac{3(b-a)}{4} \le x \le b, \\ -\mu\varepsilon, & \text{if } \frac{b-a}{4} < x < \frac{3(b-a)}{4}. \end{cases} \tag{2.14b}$$

We now have

LEMMA 2.5: *Assume that a and b are fixed, and let $z(x,t)$ be a solution of (2.13a) such that $z(x,t) \le f(t)$ in $Q = [a,b] \times [0,T)$ and $z(x,0) \le g(x)$ for $x \in [a,b]$, where f and g are given in (2.14). Then there exists $\mu > 0$ such that, for $\varepsilon > 0$ small enough, $z(x,t)$ blows up at most at $x = a, b$, at time $t = T$. More precisely, there exists a function $F(x) \equiv F(x; a, b, \varepsilon, \mu)$ such that $F(x)$ is bounded in compact subsets of (a,b), there exists $\lim_{t\uparrow T} z(x,t) = z(x,T)$ for $x \in (a,b)$, and $z(x,t) \le F(x)$ in $(a,b) \times (0,T]$.*

Proof: It is a straightforward modification of that of Proposition 3.1 in [HV2]. ∎

Notice that in Lemma 2.5, $z(x,t)$ will be negative somewhere in Q. However, only the upper bound there is nontrivial, since z is bounded below by a caloric function in Q.

2.3 FINAL-TIME ANALYSIS. We shall give here the proof of Theorem 2. To this end, we analyze the case where (1.9a) holds, and set $\bar{x} = 0$ for simplicity. We then consider the family of scaled functions

$$\begin{aligned} &v_s(x,t) = \ln(T-s) + u(\lambda(s) + x(T-s)^{1/2}, s + t(T-s)), \\ &\text{with } \lambda(s) = \xi((T-s)|\ln(T-s)|)^{1/2}, \quad \xi > 0 \text{ fixed and } 0 < s < T, \end{aligned} \tag{2.15a}$$

where

$$(T-s)^{1/2}|x| \le \frac{\lambda(s)}{2}, \quad \text{i.e. } |x| \le \frac{\xi}{2}|\ln(T-s)|^{1/2}. \tag{2.15b}$$

Taking into account (2.12a), we readily see that

$$\begin{aligned} &(v_s)_t = (v_s)_{xx} + e^{v_s} \qquad \text{for } x \in \mathbb{R}, \quad 0 < t < 1, \\ &v_s(x,0) = -\ln\left(1 + \frac{(\lambda(s) + x(T-s)^{1/2})^2}{4(T-s)|\ln(T-s)|}\right) + o(1) \qquad \text{as } s \uparrow T. \end{aligned}$$

Therefore, for any $\delta > 0$, there exists $s_0 > 0$ such that, if $s \geq s_0$,

$$-\ln\left(1+\frac{9\xi^2}{16}\right)+\delta \leq v_s(x,0) \leq -\ln\left(1+\frac{\xi^2}{16}\right)+\delta, \tag{2.16}$$

uniformly for x satisfying (2.15b) with $\xi > 0$ fixed. On the other hand, by Lemma 2.3, it follows that, for $T - s \leq 1$,

$$v_s(x,t) \leq -\ln(1-t) + \frac{C}{|\ln(T-t)|} \quad \text{for some } C > 0. \tag{2.17}$$

Set now $I_n = [-n, n]$, $n = 1, 2, \ldots$, and for any fixed n, let ε_n, μ_n, be the corresponding parameters in Lemma 2.5. Using such a result, as well as (2.16) and (2.17), we deduce that there exists a constant M_n such that

$$v_s(x,t) \leq M_n \qquad \text{in } \bar{Q} = \left[-\frac{n}{2}, \frac{n}{2}\right] \times [0,1]. \tag{2.18}$$

Notice that this implies at once that the blow-up point $x = 0$ is isolated.

We next derive a lower bound to complement (2.18). To proceed, we notice that, by (2.15), (2.16),

$$v_s(x,t) \geq w_s(x,t)$$

where

(2.19a)

$$(w_s)_t - (w_s)_{xx} = 0 \qquad \text{in } \Sigma = \left\{(x,t) : |x| < \frac{\xi}{2}|\ln(T-s)|^{1/2}, 0 < t < 1\right\},$$

(2.19b)

$$w_s(x,t) = a(s) \equiv \ln(T-s) \qquad \text{at } x = \pm\frac{\xi}{2}|\ln(T-s)|^{1/2}, \quad 0 < t < 1,$$

(2.19c)

$$w_s(x,0) = b(\xi) \equiv -\ln\left(1+\frac{9\xi^2}{16}\right)+\delta \qquad \text{for } |x| \leq \frac{\xi}{2}|\ln(T-s)|^{1/2}.$$

Let $\partial\Sigma$ denote the parabolic boundary of Σ. Then $\partial\Sigma = l_1 + l_2 + l_3$, where

$$l_1 = \{(x,t) : \; x = -\frac{\xi}{2}|\ln(T-s)|^{1/2}, 0 \leq t \leq 1\},$$

$$l_2 = \{(x,t) : \; x = \frac{\xi}{2}|\ln(T-s)|^{1/2}, 0 \leq t \leq 1\},$$

$$l_3 = \{(x,0) : \; |x| \leq \frac{\xi}{2}|\ln(T-s)|^{1/2}\}.$$

Then $w_s(x,t) = w_1(x,t) + w_2(x,t) + w_3(x,t)$, where for $i = 1,2,3$, $w_i(x,t)$ is the solution of the heat equation in $\sum$ such that $w_i(x,t) = 0$ in l_j if $i \neq j$, $w_1(x,t) = a(s)$ in l_1, $w_2(x,t) = a(s)$ in l_2, and $w_3(x,t) = b(\xi)$ in l_3. Clearly, it will suffice to derive in detail a lower bound for $w_1(x,t)$. To this end, we notice that, since we are assuming s close to T,

$$w_1(x,t) \geq z_1(x,t) \quad \text{in } \Sigma$$

where

$$\begin{aligned} &(z_1)_t - (z_1)_{xx} = 0 \qquad \text{when } x > -\frac{\xi}{2}|\ln(T-s)|^{1/2}, \quad 0<t<1,\\ &z\Big(-\frac{\xi}{2}|\ln(T-s)|^{1/2},t\Big) = \ln(T-s) \qquad \text{when } 0 \leq t \leq 1,\\ &z_1(x,0) = 0 \qquad \text{for} \quad x > -\frac{\xi}{2}|\ln(T-s)|. \end{aligned}$$

Thus, by standard results,

$$\begin{aligned} z_1(x,t) =& C\ln(T-s)\Big(x+\frac{\xi}{2}|\ln(T-s)|^{1/2}\Big)\\ &\times \int_0^t (t-s)^{-3/2}\exp\Big(-\frac{(x+\frac{\xi}{2}|\ln(T-s)|^{1/2})}{4(t-s)}\Big)ds \end{aligned}$$

where, here and henceforth, C denotes a generic constant, independent of ξ, T or s. Performing the change of variables

$$z = \frac{x+\frac{\xi}{2}|\ln(T-s)|^{1/2}}{2\sqrt{t-s}}$$

and recalling that

$$\int_y^\infty e^{-r^2}\,dr \sim \frac{e^{-y^2}}{2y}$$

as $y \to \infty$, we obtain

$$w_1(x,t) \geq z_1(x,t) \geq -\frac{C}{\xi}|\ln(T-s)|^{1/2}(T-s)^{\frac{\xi^2}{16t}}$$

which goes to zero as $s \uparrow T$ for fixed $\xi > 0$. Arguing in a similar way for w_2, w_3, we arrive at

$$\text{(2.20)} \qquad v_s(x,t) \geq -C_n \qquad \text{in} \quad \bar{Q}_n \equiv \Big[-\frac{n}{2},\frac{n}{2}\Big] \times [0,1]$$

if s is close enough to T. From (2.18) and (2.20), we deduce that there exists $K_n < +\infty$ such that

$$|v_s(x,t)| \le K_n < +\infty \qquad \text{in } \bar{Q}_n$$

whence; by Schauder estimates,

$$\left|\frac{\partial}{\partial x} v_s(x,t)\right|, \quad \left|\frac{\partial}{\partial t} v_s(x,t)\right|, \quad \left|\frac{\partial^2}{\partial x^2} v_s(x,t)\right|$$

remain bounded in the set $\bar{Q}'_{n,\delta} = [-\frac{n}{3}, \frac{n}{3}] \times [\delta, 1]$, uniformly as $s \uparrow T$, for any $\delta \in (0,1)$. It then follows that there exists a subsequence, also denoted by $\{v_s(x,t)\}$, and a function $\{\bar{v}_n(x,t)\}$, such that

(2.21a) $$v_s(x,t) \to \bar{v}_n(x,t) \quad \text{as } s \uparrow T,$$

uniformly on $\bar{Q}_{n,\delta}$ for any $\delta \in (0,1)$.

(2.21b) $$(\bar{v}_n)_t = (\bar{v}_n)_{xx} + e^{v_n} \qquad \text{in } \left(-\frac{n}{3}, \frac{n}{3}\right) \times (0,1).$$

Moreover, a classical barrier argument (cf. for instance [HV3]) shows that

(2.21c) $$\lim_{t \downarrow T} \bar{v}_n(x,t) = -\ln\left(1 + \frac{\xi^2}{4}\right), \quad \text{uniformly in } \left[-\frac{n}{4}, \frac{n}{4}\right].$$

Letting now $n \to \infty$, a standard diagonal argument shows that there exists a subsequence, again labelled as $\{v_s(x,t)\}$, and a function $\bar{v}(x,t)$ such that

(2.22a as $s \uparrow T$, uniformly on compact sets of $\mathbb{R} \times (0,1)$,) $v_s(x,t) \to \bar{v}(x,t)$

(2.22b) $$v_t = v_{xx} + e^{\bar{v}} \qquad \text{in } \mathbb{R} \times (0,1),$$

(2.22c) $$\bar{v}(x,0) = -\ln\left(1 + \frac{\xi^2}{4}\right) \quad \text{in } \mathbb{R}.$$

Furthermore, there exists $C > 0$ such that

(2.22d) $$|\bar{v}(x,t)| \le \frac{C}{\sqrt{1-t}}(1 + |x|).$$

To check (2.22d), we just remark that, arguing as in [GK1], one readily obtains that if $u(x,t)$ solves (1.7a) and blows up at $t = T$, then

$$\left|\frac{\partial u}{\partial x}(x,t)\right| \leq C(T-t)^{-1/2} \qquad \text{for some } \; C > 0,$$

which yield (2.22d) for $v_s(x,t)$, and the sought-for result follows at once by passing to the limit along the considered subsequence. We deduce from (2.22) that

$$\bar{v}(x,t) = -\ln\left(1 + \frac{\xi^2}{4} - t\right)$$

whence

$$v_s(x,t) = -\ln\left(1 + \frac{\xi^2}{4} - t\right) + o(1) \qquad \text{as } s \uparrow T \tag{2.23}$$

uniformly on compact subsets of $\mathbb{R} \times [0,1]$.

In particular, setting $x = 0$ and $t = 1$ in (2.23), we obtain

$$u\left(\xi((T-s)|\ln(T-s)|)^{1/2}, T\right) = -\ln(T-s) - \ln\frac{\xi^2}{4} + o(1) \qquad \text{as} \quad s \uparrow T. \tag{2.24}$$

If we now write $y = \xi((T-s)|\ln(T-s)|)^{1/2}$, we readily see that

$$(T-s) \simeq \frac{y^2}{2\xi^2|\ln|y||} \qquad \text{as } s \uparrow T.$$

Substituting this in (2.24), we arrive at

$$u(y,T) = -\ln\left(\frac{y^2}{8|\ln|y||}\right) + o(1) \qquad \text{as } s \uparrow T,$$

whence (1.13a) with $\bar{x} = 0$. As to (1.13b), it is obtained in a similar way: we merely replace $\lambda(s)$ in (2.15a) by $\bar{\lambda}(s) = \xi(T-s)^{1/2}$ and argue as before. ∎

2.4 THE EXISTENCE OF PLANE STRUCTURES. Let $u_R(x,t)$ be the function defined in (2.1c). It is well known that, for any fixed R, ε and h, $u_R(x,t)$ blows up in a time $T_1(R) < +\infty$. We now claim that

(2.25) There exists $M > 0$ such that $u_R(x,t) \leq M$ if $|x - R| \geq \dfrac{R}{2}$

To derive (2.25), we just notice that, by symmetry, $u_R(x,t)$ blows up at $x = R$, whereas for $t > 0$, $\frac{\partial u}{\partial x}_R(x,t) > 0$ if $x < R$ and $\frac{\partial u}{\partial x}_R(x,t) < 0$ if $x > R$. By

Theorem 2, the blow-up point $x = R$ is isolated. Therefore, if there would be another blow-up point somewhere, say at $x = R_1$, the derivative $\frac{\partial u}{\partial x}{}_R$ should change sign between $x = R$ and $x = R_1$, thus giving a contradiction.

Consider now the solution $u_R^{\pm}(x,t)$ given in (2.1d). Clearly, $u_R^{\pm}(x,t)$ blows up at a time $T(R) \leq T_1(R)$. Actually, it can be shown that $T(R) < T_1(R)$, but we do not need strict inequality here. We next show that

$$\text{For any } R > 0, \text{ there exist } \bar{M} > 0 \text{ and } D > 0 \text{ such that} \quad u_R^{\pm}(x,t) < \bar{M} \quad \text{if } |x| \geq D. \tag{2.26}$$

To show (2.26), we first assume that $T(R) < T_1(R)$ and consider the scaled function

$$v_R(x,t) = \ln\left(\frac{T(R)}{T_1(R)}\right) + v_R^{\pm}\left(\left(\frac{T(R)}{T_1(R)}\right)^{1/2} x, \frac{T(R)t}{T_1(R)}\right) \tag{2.27}$$

which blows up at $t = T_1(R)$. Set now $z = v_R - u_R$. Clearly,

$$z_t = z_{xx} + C(x,t)z \qquad \text{when} \quad x \in \mathbb{R},\ t < T_1(R)$$

where

$$C(x,t) = \frac{e^{v_R} - e^{u_R}}{v_R - u_R} \qquad \text{if } v_R \neq u_R, \quad C(x,t) \geq 0,$$

By Lemma 2.3, we have that for any fixed $s < T_1(R)$,

$$C(x,t) \leq L \qquad \text{for } x \in \mathbb{R}\ ,\ t \leq s, \text{ where } L = L(s).$$

On the other hand,

$$z(x,0) \leq \bar{z}(x,0) = v_R^{\pm}\left(\left(\frac{T(R)}{T_1(R)}\right)^{1/2} x, 0\right) - u_R(x,0),$$

and $\bar{z}(x,0)$ is compactly supported and bounded. We then conclude that for $t \leq s < T_1(R)$, there exist $D_1 > 0$ and $H > 0$ such that

$$z(x,t) \leq \frac{He^{Lt}}{(4\pi t)^{1/2}}\left(\int\limits_{|\xi - R| \leq D_1} e^{-\frac{(x-\xi)^2}{4t}} d\xi + \int\limits_{|\xi + R| \leq D_1} e^{-\frac{(x-\xi)^2}{4t}} d\xi\right) \tag{2.28a}$$

and, since $|x-\xi| \geq |x-R| - |R-\xi| \geq \frac{1}{2}|x-R|$ if $|x-R| \geq 2D_1$ and $|\xi - R| \leq D_1$, we readily check that

$$\int\limits_{|\xi - R| \leq D_1} e^{-\frac{(x-\xi)^2}{4t}} d\xi \leq 2D_1 e^{-\frac{(x-R)^2}{16t}} \tag{2.28b}$$

and arguing similarly for the second integral in (2.28a), we obtain

$$z(x,t) \le \frac{Ae^{LS}}{(4\pi t)^{1/2}}\left(e^{-\frac{(x-R)^2}{16t}} + e^{-\frac{(x+R)^2}{16t}}\right) \qquad \text{if } t \le s < T_1(R).$$

In particular, $z(x,t) \le M$ if $|x|$ is large enough. As $v_R = v_R - u_R + u_R$, this gives

(2.29) There exists $D_1 > 0$ such that $v_R(x,t) \le 2M$ if $|x| \ge D_1$.

Let now $y \in \mathbb{R}$ and $s < T_1(R)$ be given, and consider the function

$$(2.30) \qquad v_s(x,t;y) = \ln(T-s) + v_R(y + (T_1(R)-s)^{1/2},\ s + t(T_1(R)-s))$$

It is readily seen that there exists $C > 0$, independent of y and s, such that

$$(2.31a) \qquad v_s(x,t;y) \le -\ln(1-t) + \frac{C}{|\ln(T_1(R)-s)|}$$

(2.31b) $v_s(x,0;) = \ln(T_1(R)-s) + 2M$, provided that $|y|$ is large enough.

Consider now the cylinder $Q_1 = [-1,1] \times [0,1]$, and let ε, μ be the corresponding parameters in Lemma 2.5. We then deduce that, if s is close enough to $T_1(R)$, there exists $N > 0$ such that $v_s(0,1) \le N$. Recalling (2.30), we conclude that

$$v_R(y, T_1(R)) \le N - \ln(T_1(R)-s) < +\infty,$$

if $|y|$ is large enough. Taking into account (2.27), (2.26) follows.

If we assume $T(R) = T_1(R)$, the proof of (2.26) is straightforward.

As a consequence of (2.26), it follows that for any solution $v_R^{\pm}(x,t)$, blow-up occurs in a compact set (which obviously depends on R). By Theorem 2, singularities appear at isolated points, and, by symmetry, blow-up occurs at $x = 0$ or at two points $x = \pm x_0$. We now prove that

(2.32) If $R > 0$ is large enough,
$v_R^{\pm}(x,t)$ remains bounded at $x = 0$ for any $t \le T(R)$.

Actually, (2.32) follows from a minor modification of our previous argument: we just need to check that, as $R_1 \to \infty$, the constant D_1 in (2.28) remains bounded. By (2.27), this last is a consequence of the fact that $1 \ge \frac{T(R)}{T_1(R)} \ge \mu > 0$ for some constant μ, as can be seen by comparing $v_R^{\pm}(x,t)$ with the solution of (1.7a) with initial value equal to h (cf (2.1)).

Having shown (2.26) and (2.32), we now conclude as in the power-like case considered in [HV2]. It is immediate to see that, for $R = 0$, $v_R^{\pm}(x,t)$ blows up at $x = 0$ only, and the same happens if $R > 0$ is small enough. This last can be proved by using Lemma 2.2 and standard interior estimates to derive that $\frac{\partial^2 u}{\partial x^2} R(0,t) < 0$ if $t_0 < t < T(R)$ for some $t_0 > 0$. We then define

$$R_* = \sup\{R > 0 : \text{ blow-up occurs only at } x = 0\}.$$

Three possible cases arise now

(i) Collapse of maxima of $u_{R^*}(x,t)$ occurs before blow-up at $x = 0$.

(ii) There is a single-point blow-up at $x = 0$, but for any $t < T(R_1^*)$, $u_{R^*}(x,t)$ has two maxima.

(iii) blow-up takes place at two symmetric points, $x = \pm x_0$.

Case (i) is easily ruled out by means of a continuity argument (involving Lemma 2.2 as before). As to (iii), it will be excluded as a consequence of our next result:

LEMMA 2.6: *Let R_0 be such that $u_{R_0}^{\pm}(x,t)$ blows up at $x = \pm x_0$. Then, for any $\sigma \in (0, |x_0|)$, there exists $\eta > 0$ such that $u_R(x,t)$ blows up outside of the interval $[-\sigma, \sigma]$ provided that $|R - R_0| < \eta$.*

Proof: We consider again the scaled function $v_R(x,t)$ given in (2.27) with $T_1(R)$ replaced by $T(R_0)$. By assumption, $u_{R_0}(x,t) \leq M < +\infty$ if $|x| \leq \sigma_1 < |x_0|$ for some constants M and σ_1. Therefore, if $|R - R_0|$ is small enough, there exists $\sigma_1' \leq \sigma_1$ such that

$$v_R(x,t) \leq 2M \qquad \text{if } |x| \leq \sigma_1',\ t \leq 1.$$

For any given $\lambda > 0$, we now rescale as follows:

$$v_{R,\lambda}(x,t) = \ln \lambda + v_R(\sqrt{\lambda}x, 1 - \lambda + \lambda t).$$

Selecting λ small enough, we deduce from Lemma 2.5 that $v_{R,\lambda}(x,t)$ stays bounded above in a cylinder $Q^n = [-\sigma_2, \sigma_2] \times [0,1]$ for some $\sigma_2 < \sigma_1$, whence the result. ∎

END OF THE PROOF OF THEOREM 1. We have shown that, out of cases (i), (ii) and (iii) above, only (ii) can occur. Then, by our previous results, the space structure of $\psi_{R^*}(y,\tau)$ (cf (1.8)) must correspond to (1.9a) or (1.9b) with $m = 4$, since $H_m(y)$ has more maxima than ψ_{R^*} when $m \geq 6$. Assume now that $\psi_{R^*}(y,\tau)$ behaves as indicated by (1.9a).

Then $\frac{\partial^2}{\partial y^2}\psi_{R^*}(0,\tau) < 0$ for large enough τ, which contradicts the fact that $\psi_{R^*}(y,\tau)$ has a minimum at $y = 0$ by (ii). ∎

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